

# Rainbow vertex-connection number of 2-connected graphs\*

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## Abstract

The *rainbow vertex-connection number*,  $rvc(G)$ , of a connected graph  $G$  is the minimum number of colors needed to color its vertices such that every pair of vertices is connected by at least one path whose internal vertices have distinct colors. In this paper we first determine the rainbow vertex-connection number of cycle  $C_n$  of order  $n \geq 3$ , and then, based on it, prove that for any 2-connected graph  $G$ ,  $rvc(G) \leq rvc(C_n)$ , giving a tight upper bound for the rainbow vertex-connection. As a consequence, we show that for a connected graph  $G$  with a block decomposition  $B_1, B_2, \dots, B_k$  and  $t$  cut vertices,  $rvc(G) \leq rvc(B_1) + rvc(B_2) + \dots + rvc(B_k) + t$ .

**Keywords:** rainbow vertex-coloring, rainbow vertex-connection, ear decomposition, 2-connected graph.

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## 1 Introduction

All graphs in this paper are finite, undirected and simple. We follow the terminology and notation of Bondy and Murty [1].

Let  $G$  be a vertex-colored connected graph. A path of  $G$  is a *rainbow path* if its internal vertices have distinct colors. The vertex-colored graph  $G$  is called *rainbow vertex-connected* if any two vertices are connected by at least one rainbow path. The *rainbow vertex-connection number* of a connected graph  $G$ , denoted by  $rvc(G)$ , is the smallest

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number of colors that are needed in order to make  $G$  rainbow vertex-connected. If  $F$  is a subgraph of a vertex-colored graph,  $\text{colours}(F)$  denotes the set of all colors appeared in  $F$ . If a rainbow vertex-coloring of  $G$  uses  $k$  colors, we call it a  $k$ -rainbow vertex-coloring.

Let  $F$  be a subgraph of a graph  $G$ . An *ear* of  $F$  in  $G$  is a nontrivial path whose two ends are in  $F$  but whose internal vertices are not. A nested sequence of graphs is a sequence  $(G_0, G_1, \dots, G_k)$  of graphs such that  $G_i \subset G_{i+1}$ ,  $0 \leq i \leq k-1$ . An *ear decomposition* of a 2-connected graph  $G$  is a nested sequence  $(G_0, G_1, \dots, G_k)$  of 2-connected subgraphs of  $G$  satisfying the conditions: (1)  $G_0$  is a cycle; (2)  $G_i = G_{i-1} \cup P_i$ , where  $P_i$  is an ear of  $G_{i-1}$  in  $G$ ,  $1 \leq i \leq k$ ; (3)  $G_k = G$ . An ear with odd (resp. even) length is called an odd (resp. even) ear.

A maximal connected subgraph without any cut vertex is called a *block*. Thus, every block of a nontrivial connected graph is either a maximal 2-connected subgraph or  $K_2$ . All the blocks of a graph  $G$  form a *block decomposition* of  $G$ .

Given two walks  $W_1 = u_0, u_1, \dots, u_k$  and  $W_2 = v_0, v_1, \dots, v_\ell$  such that  $u_k = v_0$ , we can *concatenate*  $W_1$  and  $W_2$  to get a long walk,  $W = W_1 W_2 = u_0, u_1, \dots, u_k (= v_0), v_1, v_2, \dots, v_\ell$ . We denote the order of a graph by  $|G|$ .

The concept of rainbow vertex-connection number was introduced by Krivelevich and Yuster in [4]. Some easy observations about rainbow vertex-connection number include that if  $G$  is a connected graph of order  $n$ , then  $\text{diam}(G) - 1 \leq \text{rvc}(G) \leq n - 2$ ;  $\text{rvc}(G) = 0$  if and only if  $G$  is a complete graph and  $\text{rvc}(G) = 1$  if and only if  $\text{diam}(G) = 2$ . Krivelevich and Yuster [4] showed that if a connected graph  $G$  has  $n$  vertices and minimum degree  $\delta$ , then  $\text{rvc}(G) \leq 11n/\delta$ . In [6], Li and Shi improved the bound. In [3], Chen, Li and Shi studied the computational complexity of rainbow vertex-connection and proved that computing  $\text{rvc}(G)$  is NP-hard.

In this paper the rainbow vertex-connection  $\text{rvc}(C_n)$  of a cycle  $C_n$  ( $n \geq 3$ ) is determined. Based on it, we then prove that for any 2-connected graph  $G$  of order  $n$  ( $n \geq 3$ ),  $\text{rvc}(G) \leq \text{rvc}(C_n)$  which gives a tight upper bound for the rainbow vertex-connection of a 2-connected graph. As a consequence, we show that for a connected  $G$  with a block decomposition  $B_1, B_2, \dots, B_k$  and  $t$  cut vertices,  $\text{rvc}(G) \leq \text{rvc}(B_1) + \text{rvc}(B_2) + \dots + \text{rvc}(B_k) + t$ . The proof is constructive and hence we can give a method to construct a rainbow vertex-coloring of the connected graph  $G$  using at most  $\lceil \frac{|B_1|}{2} \rceil + \dots + \lceil \frac{|B_k|}{2} \rceil + t$  colors.

## 2 Main results

First of all, we need to introduce the concept of revised rainbow vertex-coloring.

**Definition 2.1.** Let  $G$  be a connected graph with a vertex-coloring  $c$ . A path  $P$  of  $G$

is called a *revised rainbow path* if all vertices of  $P$  have distinct colors, or all but the end vertices of  $P$  have distinct colors, which means that only the two end-vertices of  $P$  may have the same color. The vertex-coloring  $c$  of  $G$  is called a *revised rainbow vertex-coloring* if any two vertices of  $G$  are connected by at least one revised rainbow path. The *revised rainbow vertex-connection number* of a connected graph  $G$ , denoted by  $rvc^*(G)$ , is the smallest number of colors that are needed in order to make  $G$  revised rainbow vertex-connected.

Since a revised rainbow path is also a rainbow path,  $rvc(G) \leq rvc^*(G)$ . At first, we consider the rainbow vertex-connection of a cycle.

**Theorem 2.1.** *Let  $C_n$  be a cycle of order  $n(n \geq 3)$ . Then*

$$rvc(C_n) = \begin{cases} 0 & \text{if } n = 3; \\ 1 & \text{if } n = 4, 5; \\ 3 & \text{if } n = 9; \\ \lceil \frac{n}{2} \rceil - 1 & \text{if } n = 6, 7, 8, 10, 11, 12, 13, \text{ or } 15; \\ \lceil \frac{n}{2} \rceil & \text{if } n \geq 16 \text{ or } n = 14. \end{cases}$$

*Proof.* Assume that  $C_n = v_1, v_2, \dots, v_n, v_{n+1}(=v_1)(n \geq 3)$ . It is obvious that  $rvc(C_3) = 0$ . Since  $rvc(G) = 1$  if and only if  $\text{diam}(G) = 2$ , we have  $rvc(C_4) = rvc(C_5) = 1$ .

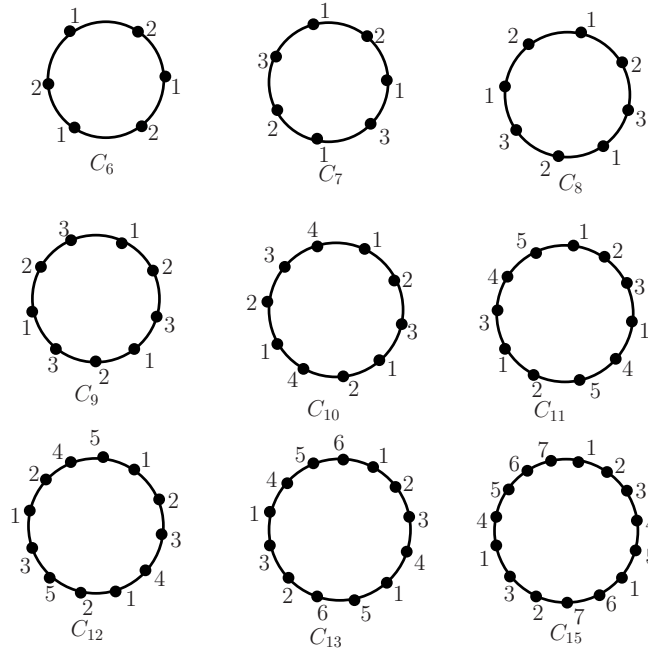


Figure 1. Rainbow vertex-colorings for small cycles.

It is easy to check that the vertex-colorings of  $C_n$  shown in Fig.1 are rainbow vertex-colorings. So  $rvc(C_n) \leq \lceil \frac{n}{2} \rceil - 1$  for  $6 \leq n \leq 13$  and  $n = 15$  and  $rvc(C_9) \leq 3$ . Since

$rvc(C_n) \geq \text{diam}(C_n) - 1$ ,  $rvc(C_n) = \lceil \frac{n}{2} \rceil - 1$  for  $n = 6, 8, 10, 12$  and  $rvc(C_9) = 3$ . For any vertex-coloring  $c$  of  $C_7$  using 2 colors, there exist two adjacent vertices (say  $v_1, v_2$ ) having the same color. Then the two paths  $P_1 = v_7, v_1, v_2, v_3$  and  $P_2 = v_7, v_6, v_5, v_4, v_3$  on  $C_n$  are not rainbow paths, i.e., there is no rainbow path between  $v_7$  and  $v_3$ . So  $c$  is not a rainbow vertex-coloring. Hence,  $rvc(C_7) = 3$ . Assume, to the contrary, that  $C_n$  ( $n = 11, 13, 15$ ) has a  $(\lceil \frac{n}{2} \rceil - 2)$ -rainbow vertex-coloring  $c$ . Then some three vertices (say  $v_1, v_i, v_j \in V(C_n), 1 < i < j \leq n$ ) have the same color and one pair of vertices among them (say  $v_1, v_i$ ) has distance no more than  $\lfloor \frac{n}{3} \rfloor$ , i.e.,  $d_{C_n}(v_1, v_i) \leq \lfloor \frac{n}{3} \rfloor$ . Let  $P = v_1, v_2, \dots, v_i$  be the path from  $v_1$  to  $v_i$  with length  $d_{C_n}(v_1, v_i)$ . Since  $c(v_1) = c(v_i)$ ,  $P_1 = v_n, v_1, \dots, v_i, v_{i+1}$  is not a rainbow path. So  $P_2 = C_n - P = v_n, v_{n-1}, \dots, v_{i+1}$  is the rainbow path from  $v_n$  to  $v_{i+1}$ . Since  $\ell(C_n - P) = n - (\ell(P) + 2) \geq n - \lfloor \frac{n}{3} \rfloor - 2 \geq \lceil \frac{n}{2} \rceil$  for  $n = 11, 13, 15$ ,  $C_n - P$  has at least  $\lceil \frac{n}{2} \rceil - 1$  internal vertices. So  $C_n - P$  is not a rainbow path, a contradiction. Hence,  $rvc(C_n) = \lceil \frac{n}{2} \rceil - 1$  for  $n = 11, 13, 15$ .

In the following, we consider the rainbow vertex-connection of  $C_n$  for  $n \geq 16$  or  $n = 14$ . Suppose  $C_n = v_1 v_2 \dots v_n v_{n+1} (= v_1)$ . Define a vertex-coloring  $c$  of  $C_n$  by  $c(v_i) = i$  for  $1 \leq i \leq \lceil \frac{n}{2} \rceil$  and  $c(v_i) = i - \lceil \frac{n}{2} \rceil$  if  $\lceil \frac{n}{2} \rceil + 1 \leq i \leq n$ . Since for any two vertices  $u, v$  of  $C_n$ , the path on  $C_n$  with length  $d_{C_n}(u, v)$  is a rainbow path, we have  $rvc(C_n) \leq \lceil \frac{n}{2} \rceil$  for  $n \geq 16$  or  $n = 14$ .

Next we show that  $rvc(C_n) \geq \lceil \frac{n}{2} \rceil$  for  $n \geq 16$  or  $n = 14$ . Assume, to the contrary, that  $rvc(C_n) \leq \lceil \frac{n}{2} \rceil - 1$ . Then there exists a  $(\lceil \frac{n}{2} \rceil - 1)$ -rainbow vertex-coloring  $c_0$  of  $C_n$ . Obviously, there are three vertices (say  $v_1, v_i, v_j, 1 \leq i < j \leq n$ ) of  $C_n$  having the same color. And one pair of vertices among  $\{v_1, v_i, v_j\}$  (say  $v_1, v_i$ ) satisfies that  $d_{C_n}(v_1, v_i) \leq \lfloor \frac{n}{3} \rfloor$ . Without loss of generality, assume that  $P = v_1, v_2, \dots, v_i$  is the path on  $C_n$  with length  $d_{C_n}(v_1, v_i)$ . Now consider the vertices  $v_n$  and  $v_{i+1}$ . Since  $v_1$  and  $v_i$  have the same color, the rainbow path between  $v_n$  and  $v_{i+1}$  on  $C_n$  must be  $C_n - P$ . Since  $\ell(C_n - P) = n - (\ell(P) + 2) \geq n - \lfloor \frac{n}{3} \rfloor - 2$ , the number of internal vertices of  $C_n - P$  is at least  $n - \lfloor \frac{n}{3} \rfloor - 3$ . For  $n \geq 16$  or  $n = 14$ ,  $n - \lfloor \frac{n}{3} \rfloor - 3 > \lceil \frac{n}{2} \rceil - 1$  which contradicts that  $C_n - P$  is a rainbow path. Hence  $rvc(C_n) \geq \lceil \frac{n}{2} \rceil$  for  $n \geq 16$  or  $n = 14$ . Therefore,  $rvc(C_n) = \lceil \frac{n}{2} \rceil$  for  $n \geq 16$  or  $n = 14$ .  $\square$

Before proceeding, we introduce another notion.

**Definition 2.2.** (Balanced coloring) Let  $H$  be a connected graph and  $P$  an ear of  $H$  such that  $V(H) \cap V(P) = \{a, b\}$ . Assume that  $P = v_1 (= a), v_2, \dots, v_s (= b) (s \geq 6)$  and  $H$  has a vertex-coloring  $c'$ . Let  $x$  be a color of  $c'$  and  $x_1, x_2, \dots, x_{\lceil \frac{s}{2} \rceil - 1}$  new colors. Now define a vertex-coloring  $c$  of  $G := H \cup P$  from  $c'$ . We distinguish the following cases according to the parities of  $|H|$  and  $\ell(P)$ .

Case 1.  $|H|$  is even and  $\ell(P) (= s - 1)$  is odd.

$c(v) = c'(v)$  for  $v \in V(H) \setminus \{a, b\}$ ; the last  $\frac{s}{2}$  vertices of  $P$ , i.e.,  $v_{\frac{s}{2}+1}, \dots, v_s$  are colored by  $c'(b), x_1, \dots, x_{\frac{s}{2}-1}$  in order, and if  $c'(a) \neq c'(b)$ , then the first  $\frac{s}{2}$  vertices of  $P$ , i.e.,  $v_1, \dots, v_{\frac{s}{2}}$  are colored by  $x_1, \dots, x_{\frac{s}{2}-1}, c'(a)$  in order; otherwise, by  $c'(a), x_1, \dots, x_{\frac{s}{2}-1}$  in order.

Case 2.  $|H|$  is odd and  $\ell(P)$  is even.

$c(v) = c'(v)$  for  $v \in V(H) \setminus \{a, b\}$ ; the last  $\lceil \frac{s}{2} \rceil - 1$  vertices of  $P$ , i.e.,  $v_{\lceil \frac{s}{2} \rceil+1}, \dots, v_s$  are colored by  $c'(b), x_1, \dots, x_{\lceil \frac{s}{2} \rceil-2}$  in order, and if  $c'(a) \neq c'(b)$ , then the first  $\lceil \frac{s}{2} \rceil$  vertices of  $P$ , i.e.,  $v_1, \dots, v_{\lceil \frac{s}{2} \rceil}$  are colored by  $x_1, \dots, x_{\lceil \frac{s}{2} \rceil-2}, c'(a), x$  in order; otherwise, by  $c'(a), x_1, \dots, x_{\lceil \frac{s}{2} \rceil-2}, x$  in order.

Case 3.  $|H|$  and  $\ell(P)$  are even.

First, color one vertex of  $P$  by  $x_{\lceil \frac{s}{2} \rceil-1}$ . Second,  $c(v) = c'(v)$  for  $v \in V(H) \setminus \{a, b\}$ ; the last  $\lceil \frac{s}{2} \rceil - 1$  uncolored vertices of  $P$  are colored by  $c'(b), x_1, \dots, x_{\lceil \frac{s}{2} \rceil-2}$  in order, and if  $c'(a) \neq c'(b)$ , then the first  $\lceil \frac{s}{2} \rceil - 1$  uncolored vertices of  $P$  are colored by  $x_1, \dots, x_{\lceil \frac{s}{2} \rceil-2}, c'(a)$  in order; otherwise, by  $c'(a), x_1, \dots, x_{\lceil \frac{s}{2} \rceil-2}$  in order.

Case 4.  $|H|$  and  $\ell(P)$  are odd.

First, color one vertex of  $P$  by  $x_{\frac{s}{2}-1}$ . Second,  $c(v) = c'(v)$  for  $v \in V(H) \setminus \{a, b\}$ ; the last  $\frac{s}{2} - 1$  uncolored vertices of  $P$  are colored by  $c'(b), x_1, \dots, x_{\frac{s}{2}-2}$  in order, and if  $c'(a) \neq c'(b)$ , then the first  $\frac{s}{2}$  uncolored vertices of  $P$  are colored by  $x_1, \dots, x_{\frac{s}{2}-2}, c'(a), x$  in order; otherwise, by  $c'(a), x_1, \dots, x_{\frac{s}{2}-2}, x$  in order.

The obtained vertex-coloring  $c$  of  $G$  from  $c'$  is called a *balanced coloring*.  $\square$

It is obvious that when  $|G|$  is even, the balanced coloring  $c$  is unique with respect to  $c'$ , and when  $|G|$  is odd, it is not unique.

Let  $G$  be a connected graph and  $v$  a vertex of  $G$ . Assume that  $c$  is a revised rainbow vertex-coloring of  $G$  and  $x$  is a color of  $c$  such that  $c(v) \neq x$ . Define a *property*  $(*)$  of  $c$  with respect to  $v$  and  $x$  as follows: for any vertex  $u$  of  $G$  which  $c(u) \neq x$ , there exists a revised rainbow path  $P$  from  $v$  to  $u$  such that  $x \notin \text{colours}(P)$ .

**Proposition 2.1.** Let  $H$  be a connected graph and  $P = v_1(=a), v_2, \dots, v_s(=b) (s \geq 6)$  an ear of  $H$  such that  $V(H) \cap V(P) = \{a, b\}$ .  $H$  has a revised  $\lceil \frac{|H|}{2} \rceil$ -rainbow vertex-coloring  $c'$  each of whose colors appears at most twice. Moreover, when  $|H|$  is odd,  $c'$  satisfies the property  $(*)$  with respect to  $a$  and  $x'$  ( $x'$  is the color appeared once in  $c'$  and  $c'(a) \neq x'$ ). In Cases 2 and 4 of Definition 2.2, we choose  $x$  as the color  $x'$ . Then we have:

(a) The balanced coloring  $c$  of  $G := H \cup P$  from  $c'$  is a revised  $\lceil \frac{|G|}{2} \rceil$ -rainbow vertex-coloring such that every color appears at most twice.

(b) When  $|G|$  is odd, for any vertex  $v \in V(G)$  there exists a balanced coloring  $c$  of  $G$  satisfying the property  $(*)$  with respect to  $v$  and  $x_{\lceil s/2 \rceil-1}$  ( $x_{\lceil s/2 \rceil-1}$  is the color appeared

once in  $c$ ).

*Proof.* We first prove (a). From the definition of balanced coloring and the properties of  $c'$ , the balanced coloring  $c$  uses  $\lceil \frac{|G|}{2} \rceil$  colors and every color appears at most twice. Since  $c'$  is a revised rainbow vertex-coloring of  $H$ ,  $H$  is revised rainbow vertex-connected with respect to the balanced coloring  $c$ . Hence, for any two vertices in  $V(H)$  there exists a revised rainbow path in  $H$  with respect to  $c$ . Consider two vertices  $v_1 \in V(H)$  and  $v \in V(P)$ . Let  $P_a$  (resp.  $P_b$ ) be a revised rainbow path from  $v_1$  to  $a$  (resp.  $b$ ) in  $H$  such that  $x' \notin \text{colours}(P_a)$  if  $|H|$  is odd and  $c(v_1) \neq x'$ . Such a revised rainbow path  $P_a$  exists since  $c'$  satisfies the property (\*) with respect to  $a$  and  $x'$  when  $|H|$  is odd. Then one of  $P_a(aPv_2)$  and  $P_b(bPv_2)$  is a revised rainbow path from  $v_1$  to  $v_2$ . Note that when  $|H|$  is odd, the vertex on  $P$  colored by  $x'$  has a revised rainbow path to any vertex in  $V(H)$  since  $c'$  satisfies the property (\*) with respect to  $a$  and  $x'$ . Let  $P'$  be a revised rainbow path from  $a$  to  $b$  in  $H$ . Then for any two vertices in  $V(P)$  there exists a revised rainbow path between them on the cycle  $P \cup P'$ . Therefore, the balanced coloring  $c$  of  $G$  is a revised  $\lceil \frac{|G|}{2} \rceil$ -rainbow vertex-coloring such that every color appears at most twice.

Now we prove (b). It is obvious that  $|G|$  is odd in Cases 3 and 4 of Definition 2.2. First, consider  $v \in V(G) \setminus \{a, b\}$ . In Case 3 of Definition 2.2, assign the color  $x_{\lceil s/2 \rceil - 1}$  to the vertex  $v_{\lceil s/2 \rceil}$ . In Case 4 of Definition 2.2, if  $c(v) \neq c'(a)$ , then assign the color  $x_{s/2 - 1}$  to  $v_{s/2 + 1}$ ; otherwise, we have  $c(v) \neq c'(b)$  and assign the color  $x_{s/2 - 1}$  to  $v_{s/2}$ . The other vertices are colored according to Definition 2.2. From (a), the obtained balanced coloring  $c$  is a revised rainbow vertex-coloring. Let  $P_a$  (resp.  $P_b$ ) be a revised rainbow path from  $v$  to  $a$  (resp.  $b$ ) such that  $x' \notin \text{colours}(P_a)$  if  $|H|$  is odd and  $c(v) \neq x'$ . Then for any vertex  $v' \in V(H)$ , the revised rainbow path  $P'$  from  $v$  to  $v'$  in  $H$  satisfies that  $x_{\lceil s/2 \rceil - 1} \notin \text{colours}(P')$ . For any vertex  $v' \in V(P)$  ( $c(v') \neq x_{\lceil s/2 \rceil - 1}$ ), one of  $P_a(aPv_2)$  and  $P_b(bPv_2)$  is a revised rainbow path from  $v$  to  $v'$  such that  $x_{\lceil s/2 \rceil - 1}$  does not appear on the path. Second, consider  $v \in V(P)$ . Assign the color  $x_{\lceil s/2 \rceil - 1}$  to the vertex  $u \in V(P)$  such that  $d_P(u, v) = \lceil s/2 \rceil$  in Definition 2.2. It can be checked that the obtained balanced coloring of  $G$  has the property (\*) with respect to  $v$  and  $x_{\lceil s/2 \rceil - 1}$ , i.e., for any vertex  $v' \neq u$ , there exists a revised rainbow path  $P$  from  $v$  to  $v'$  such that  $u \notin V(P)$ . It is clear that the color  $x_{\lceil s/2 \rceil - 1}$  appears once in  $c$ .  $\square$

Let  $G$  be a 2-connected graph of order  $n$  ( $n \geq 3$ ). Then  $G$  has a nonincreasing ear decomposition  $(G_0, G_1, \dots, G_k)$  satisfying the following conditions: (i)  $G_0$  is an even cycle, if  $G$  is not an odd cycle; (ii)  $G_i = G_{i-1} \cup P_i$  ( $1 \leq i \leq k$ ) where  $P_i$  is a longest ear of  $G_{i-1}$ , i.e.,  $\ell(P_1) \geq \ell(P_2) \geq \dots \geq \ell(P_k)$ ; (iii)  $V(P_i) \cap V(G_{i-1}) = \{a_i, b_i\}$  ( $1 \leq i \leq k$ ) such that  $a_i \neq b_i$ . In the sequel all the nonincreasing ear decomposition of a 2-connected graph is one defined as above and the order of  $G_i$  is denoted by  $n_i$ . Without loss of

generality, assume that  $\ell(P_t) \geq 2$  and  $\ell(P_{t+1}) = \dots = \ell(P_k) = 1$ . Hence,  $G_t$  is a minimal 2-connected spanning subgraph of  $G$ , and every 2-connected graph has such a spanning subgraph. It is obvious that  $rvc(G_t) \leq rvc(G)$ . Therefore, in order to give an upper bound for the rainbow vertex-connection of 2-connected graphs, we just need to consider minimal 2-connected graphs.

**Lemma 2.1.** Let  $G$  be a 2-connected graph of order  $n$  ( $n \geq 16$ ). If the nonincreasing ear decomposition  $(G_0, G_1, \dots, G_t)$  of  $G$  satisfies that  $\ell(P_1) \geq \dots \geq \ell(P_t) \geq 5$  (if  $t \geq 1$ ), then  $G$  has a revised  $\lceil \frac{n}{2} \rceil$ -rainbow vertex-coloring such that every color appears at most twice, i.e.,  $rvc(G) \leq rvc^*(G) \leq \lceil \frac{n}{2} \rceil$ .

*Proof.* We prove the lemma by demonstrating a revised  $\lceil \frac{n}{2} \rceil$ -rainbow vertex-coloring of  $G$  such that every color appears at most twice. If  $G$  is an odd cycle, i.e.,  $G = C_n = v_1, v_2, \dots, v_n, v_{n+1}(= v_1)$ , then define a vertex-coloring of  $G$  by  $c(v_i) = x_i$  for  $1 \leq i \leq \lceil n/2 \rceil$  and  $c(v_i) = x_{i-\lceil n/2 \rceil}$  if  $\lceil n/2 \rceil + 1 \leq i \leq n$ . Since for any two vertices  $u, v$  of  $G$  the path  $P$  between  $u$  and  $v$  with length  $d_G(u, v)$  is a revised rainbow path,  $c$  is a revised  $\lceil \frac{n}{2} \rceil$ -rainbow vertex-coloring such that every color appears at most twice.

In the following, we assume that  $G$  is not an odd cycle. We will apply Proposition 2.1 to show that  $G_i$  ( $0 \leq i \leq t$ ) has a revised  $\lceil \frac{n_i}{2} \rceil$ -rainbow coloring  $c_i$  such that every color appears at most twice. Assume that  $G_0 = C_{n_0} = v_1, v_2, \dots, v_{n_0}, v_{n_0+1}(= v_1)$ . Define a vertex-coloring  $c_0$  of  $G_0$  by  $c_0(v_i) = x_i$  for  $1 \leq i \leq n_0/2$  and  $c_0(v_i) = x_{i-n_0/2}$  if  $n_0/2 + 1 \leq i \leq n_0$ . It can be checked that  $c_0$  is a revised  $\frac{n_0}{2}$ -rainbow vertex-coloring such that every color appears twice. If  $t = 0$ , the result holds. Assume that  $t > 0$ . Since the vertex-coloring of  $c_0$  satisfies the conditions of the Proposition 2.1,  $G_1 = G_0 \cup P_1$  has a balanced coloring  $c_1$  from  $c_0$  satisfying the properties: (a)  $c_1$  is a revised  $\lceil \frac{n_1}{2} \rceil$ -rainbow vertex-coloring such that every color appears at most twice; (b) when  $n_1$  is odd and  $t > 1$ ,  $c_1$  has the property (\*) with respect to  $a_2$  and  $x'_1$  ( $x'_1$  is the color appeared once in  $c_1$ ). If  $t = 1$ , the result holds. Consider the case that  $t \geq 2$ . It is obvious that the balanced coloring  $c_1$  of  $G_1$  satisfies the conditions of Proposition 2.1. Hence, using Proposition 2.1  $t$  times we can obtain a balanced coloring  $c_i$  of  $G_i$  ( $1 \leq i \leq t$ ) from  $c_{i-1}$  satisfying the properties: (a)  $c_i$  is a revised  $\lceil \frac{n_i}{2} \rceil$ -rainbow vertex-coloring such that every color appears at most twice; (b) when  $n_i$  is odd and  $i < t$ ,  $c_i$  has the property (\*) with respect to  $a_{i+1}$  and  $x'_i$  ( $x'_i$  is the color appeared once in  $c_i$ ). Therefore, we obtain a revised  $\lceil \frac{n}{2} \rceil$ -rainbow vertex-coloring  $c_t$  of  $G$  such that every color appears at most twice.  $\square$

**Theorem 2.2.** *Let  $G$  be a 2-connected graph of order  $n(n \geq 3)$ . Then*

$$rvc(G) \leq \begin{cases} 0 & \text{if } n = 3; \\ 1 & \text{if } n = 4, 5; \\ 3 & \text{if } n = 9; \\ \lceil \frac{n}{2} \rceil - 1 & \text{if } n = 6, 7, 8, 10, 11, 12, 13 \text{ or } 15; \\ \lceil \frac{n}{2} \rceil & \text{if } n \geq 16 \text{ or } n = 14, \end{cases}$$

*and the upper bound is tight, which is achieved by the cycle  $C_n$ .*

*Proof.* For  $3 \leq n \leq 15$ , it can be checked that  $rvc(G) \leq rvc(C_n)$ , i.e., the result holds from Theorem 2.1. In the following, we show that  $rvc(G) \leq \lceil \frac{n}{2} \rceil$  for  $n \geq 16$ . Without loss of generality, assume that  $G$  is a minimal 2-connected graph. So the nonincreasing ear decomposition  $(G_0, G_1, \dots, G_k)$  of  $G$  satisfies that  $\ell(P_1) \geq \dots \geq \ell(P_k) \geq 2$  if  $k \geq 1$ . If  $k = 0$  or  $\ell(P_1) \geq \dots \geq \ell(P_k) \geq 5$ , then  $G$  has a revised  $\lceil \frac{n}{2} \rceil$ -rainbow vertex-coloring from Lemma 2.1. Hence,  $rvc(G) \leq rvc^*(G) \leq \lceil \frac{n}{2} \rceil$ .

Now assume that  $k \geq 1$ ,  $5 \leq \ell(P_t) \leq \dots \leq \ell(P_1)$  and  $2 \leq \ell(P_k) \leq \dots \leq \ell(P_{t+1}) \leq 4$  ( $t < k$ ). From Proposition 2.1,  $G_t$  has a revised  $\lceil \frac{n_t}{2} \rceil$ -rainbow vertex-coloring  $c_t$  such that every color appears at most twice. Let  $x$  be a color of  $c_t$  and  $x_j (t+1 \leq j \leq k)$  and  $x_0$  are new colors.

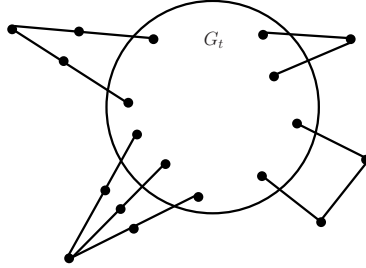


Figure 2. The graph used in the proof of Theorem 2.2.

The graph  $G$  is shown in Fig. 2 where the ears exist possibly. Define a vertex-coloring  $c$  of  $G$  from  $c_t$  as follows. For any  $v \in V(G_t)$ ,  $c(v) = c_t(v)$ . If there exists only one ear, say  $P_j = a_j, v_{j_1}, v_{j_2}, v_{j_3}, b_j (j = t+1)$  with length 4, then  $c(a_j) = c(v_{j_3}) = x_j$ ,  $c(v_{j_1}) = c_t(a_j)$  and  $c(v_{j_2}) = x$ . If there exist at least two ears with length 4 and  $P_j = a_j, v_{j_1}, v_{j_2}, v_{j_3}, b_j (t+1 \leq j \leq k)$  is such an ear with length 4, then  $c(a_j) = c(v_{j_3}) = x_j$ ,  $c(v_{j_1}) = c_t(a_j)$  and  $c(v_{j_2}) = x_0$ . Note that the center vertices of all ears with length 4 are colored by the new color  $x_0$ . If  $P_j = a_j, v_{j_1}, v_{j_2}, b_j (t+1 \leq j \leq k)$  with length 3, then  $c(a_j) = c(v_{j_2}) = x_j$  and  $c(v_{j_1}) = c_t(a_j)$ . If  $P_j = a_j, v_{j_1}, b_j (t+1 \leq j \leq k)$  with length 2, then  $c(v_{j_1}) = x$ . Note that in the definition of coloring  $c$  of  $G$  above, some vertex



$a_j(t+1 \leq j \leq k)$  is possibly colored more than once. If possible, we choose the new color  $x_j(t+1 \leq j \leq k)$  as its color. Hence, we obtain a vertex-coloring  $c$  of  $G$  from  $c_t$ .

It is clear that  $c$  uses at most  $\lceil \frac{n}{2} \rceil$  colors. In the following, we show that  $G$  is rainbow vertex-connected. Since  $c_t$  is a revised rainbow coloring of  $G_t$ ,  $G_t$  is revised rainbow vertex-connected respect to the coloring  $c$ . Hence, for any two vertices in  $V(G_t)$  there exists a revised rainbow path between them in  $G_t$ . For any two vertices  $v_1 \in V(P_j) \setminus V(G_t)$  ( $t+1 \leq j \leq k$ ) and  $v_2 \in V(G_t)$ , one of  $(v_1 P_j a_j)P'$  and  $(v_1 P_j b_j)P''$  where  $P'$  (resp.  $P''$ ) is a revised rainbow path from  $a_j$  (resp.  $b_j$ ) to  $v_2$  in  $G_t$  is a rainbow path from  $v_1$  to  $v_2$ . Consider two vertices  $v_1, v_2 \in V(G) \setminus V(G_t)$ . If  $d_G(v_1, v_2) \leq 2$ , then there is a rainbow path with length no more than 2 from  $v_1$  to  $v_2$  trivially. If  $d_G(v_1, v_2) \geq 3$ , assume that  $v_1 \in V(P_{j_1})$  and  $v_2 \in V(P_{j_2})$  ( $t+1 \leq j_1 < j_2 \leq k$ ). Since  $\ell(P_{j_2}) \leq 4$ , one of  $a_{j_2} P_{j_2} v_2$  and  $b_{j_2} P_{j_2} v_2$  (say  $a_{j_2} P_{j_2} v_2$ ) has length no more than 2. Let  $P'_{j_1}$  (resp.  $P''_{j_1}$ ) be a revised rainbow path from  $a_{j_1}$  (resp.  $b_{j_1}$ ) to  $a_{j_2}$  in  $G_t$ . Then one of  $(v_1 P_{j_1} a_{j_1})P'_{j_1}(a_{j_2} P_{j_2} v_2)$  and  $(v_1 P_{j_1} b_{j_1})P''_{j_1}(a_{j_2} P_{j_2} v_2)$  is a rainbow path from  $v_1$  to  $v_2$ . Therefore,  $c$  is a rainbow coloring of  $G$ , i.e.,  $rvc(G) \leq \lceil \frac{n}{2} \rceil$ .

From Theorem 2.1, the upper bound is tight.  $\square$

**Theorem 2.3.** *Let  $G$  be a connected graph. If  $G$  has a block decomposition  $B_1, B_2, \dots, B_k$  and  $t$  cut vertices, then  $rvc(G) \leq rvc(B_1) + rvc(B_2) + \dots + rvc(B_k) + t$ .*

*Proof.* If  $G$  is a complete graph, then  $rvc(G) = 0$  and the result holds trivially. If  $G$  is not a complete graph, then  $rvc(G) \geq 1$ . We prove the result by demonstrating a rainbow vertex-coloring using at most  $\sum_{i=1}^k rvc(B_i) + t$  colors. If all the blocks are complete graphs, then define a vertex-coloring  $c$  of  $G$  as follows. The  $t$  cut vertices are colored by  $t$  colors  $x_1, x_2, \dots, x_t$ , and other vertices are colored by  $x_1$ . In this case, for any two vertices there exists a path  $P$  between them whose interval vertices are cut vertices, i.e.,  $P$  is a rainbow path. Hence  $c$  is a rainbow vertex-coloring. The result holds.

Now assume that  $B_1, \dots, B_s$  ( $s \geq 1$ ) are not complete and  $B_{s+1}, \dots, B_k$  are complete. Then there exists a rainbow vertex-coloring  $c_i$  of  $B_i$  ( $1 \leq i \leq s$ ) using  $rvc(B_i)$  colors such that  $colours(B_i) \cap colours(B_j) = \emptyset$  ( $1 \leq i < j \leq s$ ). Define a rainbow vertex-coloring  $c_i$  of  $B_i$  ( $s+1 \leq i \leq k$ ) that every vertex of  $B_i$  is colored by a color appeared in  $B_1$ . We define a vertex-coloring  $c$  of  $G$  as follows. For any  $v \in V(B_i)$  ( $1 \leq i \leq k$ ) which is not a cut vertex,  $c(v) = c_i(v)$  and the  $t$  cut vertices are colored distinct with  $t$  new colors. It is easy to check that the vertex-coloring  $c$  of  $G$  using at most  $\sum_{i=1}^k rvc(B_i) + t$  colors is a rainbow vertex-coloring. Therefore,  $rvc(G) \leq rvc(B_1) + rvc(B_2) + \dots + rvc(B_k) + t$ .  $\square$

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